Recent advances in open billiards with some open problems

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Abstract

Much recent interest has focused on "open" dynamical systems, in which a classical map or flow is considered only until the trajectory reaches a "hole", at which the dynamics is no longer considered. Here we consider questions pertaining to the survival probability as a function of time, given an initial measure on phase space. We focus on the case of billiard dynamics, namely that of a point particle moving with constant velocity except for mirror-like reflections at the boundary, and give a number of recent results, physical applications and open problems.

1 Introduction

A mathematical billiard is a dynamical system in which a point particle moves with constant speed in a straight line in a compact domain $\mathcal{D} \subset \mathbb{R}^d$ with a piecewise smooth[1] boundary $\partial\mathcal{D}$ and making mirror-like[2] reflections whenever it reaches the boundary. We can assume that the speed and mass are both equal to unity. In some cases it is convenient to use periodic boundary conditions with obstacles in the interior, so $\mathbb{R}^d$ is replaced by the torus $\mathbb{T}^d$. Here we will mostly consider the planar case $d = 2$. Billiards are of interest in mathematics as examples of many different dynamical behaviours (see Fig. 1 below) and in physics as models in statistical mechanics and the classical limit of waves moving in homogeneous cavities; more details for both mathematics and physics are given below. An effort has been made to keep the discussion as non-technical and self-contained as possible; for further definitions and discussion, please see [20, 28, 37]. It should also be noted that the references contain only a personal and very incomplete selection of the huge literature relevant to open billiards, and that further open problems may be found in the recent reviews [13, 28, 55, 64, 66]

The structure of this article is as follows. In Secs. 2 and 3 we consider general work on closed and open dynamical systems respectively. In Sec. 4 we consider each of the main classes of billiard dynamics, and in Sec. 5 we consider physical applications. Sec. 6 returns to a more general discussion and outlook.

2 Closed dynamical systems

In this section we introduce some notation for mathematical billiards, as well as informal descriptions of properties used to characterise chaos in billiards and more general systems, with pointers to more precise formulations in the literature. Much relevant work on open systems applies to chaotic maps different from billiards, so this notation is designed to be applicable both to billiards and to more general dynamical systems. A readable introduction to the subject of (closed) billiards is given in [67] with more details of chaotic billiards in [20].

Below, $|\cdot|$ will denote the size of a set, using the Lebesgue measure of the appropriate dimension.[3] We denote the dynamics by $\Phi^t : \Omega \to \Omega$ for either flows (including the case of billiards) or

[1] The amount of smoothness required depends on the context; an early result of Lazutkin [50] required 553 continuous derivatives, but recent theorems for chaotic billiards typically require three continuous derivatives except at a small set of singular points [17] while polygonal billiards and most explicit examples are piecewise analytic. For an attempt in a (non-smooth) fractal direction, see [49].

[2] That is, the angle of incidence is equal to the angle of reflection. Popular alternative reflection laws include that of outer billiards, also called dual billiards, in which the dynamics is external to a convex domain, see for example [34], and Andreev billiards used to model superconductors [26]; we do not consider these here.

[3] All sets are assumed to be measurable with respect to the relevant measure(s).

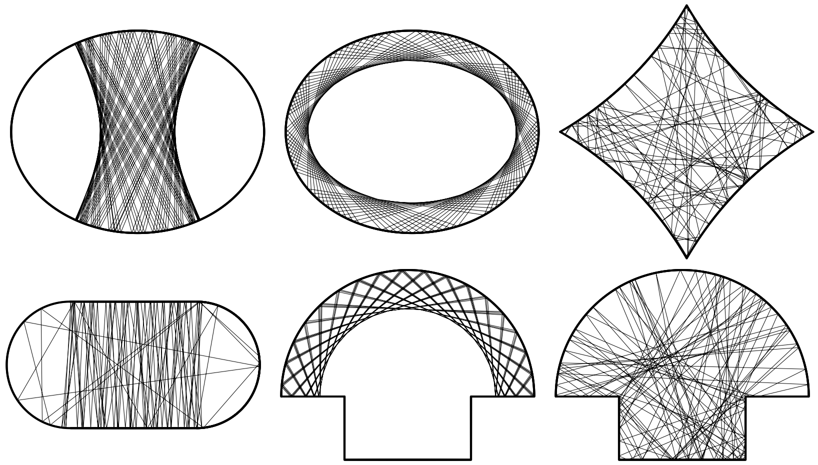

Figure 1: Geometry vs. dynamics. An integrable elliptical billiard (top left and centre) has two types of regular motion depending on the initial condition, a dispersing diamond (top right) is chaotic, a defocusing stadium (bottom left) is intermittent, with orbits switching between regular and chaotic motion, and a mushroom (bottom centre and right) is mixed, with regular or chaotic motion depending on the initial condition. Apart from the ellipse, all these are constructed from circular arcs and straight lines; many other examples with more subtle dynamical distinctions exist. See Sec. 4 for discussion of all these cases.

maps. For flows we have $t \in \mathbb{R}$ while for maps we have $t \in \mathbb{Z}$. Φ^t also naturally acts on subsets $A \subseteq \Omega$. For non-invertible maps and $t < 0$, action on points is undefined, but action on sets gives the relevant pre-image. The phase space Ω for the billiard flow consists of all[4] particle positions $q \in \mathcal{D}$ and momentum directions $p \in \mathbb{S}^{d-1}$ except that we need an extra condition to make Φ^t single-valued at collisions: For definiteness assume that p points toward the interior of $\mathcal{D}$ if q is at the boundary.

For many calculations it is more convenient to put a Poincaré section on the billiard boundary (or other convenient hypersurface for general flows), thus considering the "Birkhoff" map $F^n : M \to M$ describing $n \in \mathbb{Z}$ collisions, and acting on a reduced phase space M consisting of points $x \in \partial\mathcal{D}$ and inward pointing p. The map and flow are related by the "roof function", the time $\tau : M \to \mathbb{R}^+$ to the next collision, so that $F(x) = \Phi^{\tau(x)}(x)$ for $x \in M$.

We now mention a few[5] properties used to describe the chaoticity of dynamical systems; more details including subtle differences between flows and maps can be found in [20]. A hyperbolic map is one where the Lyapunov exponents (exponential expansion and contraction rates of infinitesimal perturbations) λ_i are all non-zero almost everywhere, and in the flow case there is a single zero exponent associated with the flow direction. For a uniformly hyperbolic system the relevant statement is true everywhere, with uniform bounds on the exponents and associated constants. For smooth uniformly hyperbolic (Anosov) systems the dynamics is controlled by unstable and stable manifolds in phase space corresponding to the positive and negative Lyapunov exponents respectively, yielding Sinai-Ruelle-Bowen (SRB) invariant measures smooth in the unstable directions, and dividing phase space into convenient entities called Markov partitions. Hyperbolicity that is nonuniform and/or nonsmooth has similar properties, but requires more general and detailed techniques [9].

[4] Billiard dynamics cannot usually be continued uniquely if the particle reaches a corner, so strictly speaking we need to exclude a zero measure set of points that do this at some time in the past and/or future. Another barrier to defining Φ^t is where there are infinitely many collisions in finite time; [20] states conditions under which this is impossible, roughly that for $d = 2$ there are finitely many corners, and focusing (convex) parts of the boundary are sufficiently (C^3) smooth, have non-zero curvature, and do not end in a cusp (zero angle corner). This is the case for almost all billiards considered in the past; exceptions may be interesting to consider in the future.

[5] The properties mentioned are those commonly discussed in the literature of open dynamical systems including billiards, and tend to be measure theoretic rather than topological.

For ergodic properties we choose an invariant measure[6] μ. For billiards there are natural "equilibrium" invariant probability measures μ_Ω and μ_M. μ_Ω is given by the usual (Lebesgue) measure on the phase space Ω, and μ_M is given by the product of measure on the boundary and the components of momentum parallel to the boundary. For example, in two dimensions $d\mu_M = ds\ dp_\parallel/(2|\partial\mathcal{D}|) = \cos\psi\ ds\ d\psi/(2|\partial\mathcal{D}|)$, where s is arc length, $p_\parallel$ is momentum parallel to the boundary and ψ is angle of incidence, that is, the angle between the momentum following a collision and the inward normal to the boundary. For integrable billiards such as the ellipse of Fig. 1, this is one of many smooth invariant measures defined by arbitrary smooth functions on the level sets of the conserved quantity, while for ergodic billiards such as the stadium or diamond of Fig. 1, it is the only smooth invariant measure.

Given an invariant measure μ, recurrence is the statement that μ-almost all trajectories return arbitrarily close to their initial point; it is guaranteed by the Poincaré recurrence theorem if μ is finite as above. Ergodicity is the statement that all invariant sets have zero or full measure, which implies that time averages of an integrable phase function are for almost all (with respect to μ) initial conditions given by its phase space average over μ. Mixing is the statement that the μ-probability of visiting region A at time zero and B at time t are statistically independent in the limit $|t| \to \infty$ for sets $A, B \subset \Omega$. There is an equivalent statement in terms of decay of correlation functions such as found in Eq. (9). Stronger ergodic properties are Kolmogorov mixing (K-mixing) and Bernoulli. We have Bernoulli $\Rightarrow$ K-mixing $\Rightarrow$ mixing $\Rightarrow$ ergodicity $\Rightarrow$ recurrence.

Further statistical properties build on the above. These include rates of decay of correlation functions for mixing systems and moderately regular (typically Hölder continuous) phase functions and central limit theorems for time averages of phase functions. See [7, 22] for a discussion of recent results in this direction.

Finally, the Kolmogorov-Sinai (KS) entropy h_{KS} is a measure of the unpredictability of the system. For a closed, sufficiently smooth map with a smooth invariant measure it is equal to the sum of the positive Lyapunov exponents (if any); this is called Pesin's formula, and has been proved for some systems with singularities including billiards. All K-mixing systems have $h_{KS} > 0$.

Note that many of these descriptors of chaos are logically independent. In the case of billiards, open problem 6 conjectures that polygonal billiards may be mixing, but they are certainly not hyperbolic, having zero Lyapunov exponents everywhere. The recently proposed "track billiards" [14] are hyperbolic but not ergodic or mixing. The stadium of Fig. 1 and the section below on defocusing billiards is Bernoulli [20] but has slow decay of correlations [7] and a non-standard central limit theorem [6].

3 Open dynamical systems

Open dynamical systems are reviewed in both the mathematical [28] and physical [2] literature. Most of the mathematical studies of open dynamical systems have considered strongly chaotic systems, such as piecewise expanding maps of the interval or Anosov (uniformly hyperbolic) maps in higher dimensions. Note that there are a variety of conventions used in the literature to describe open systems.

An open dynamical system contains a "hole" $H \subset \Omega$ at which the particle is absorbed and no longer considered. H may have more than one connected component (several "holes"), and may be on the boundary (ie a subset of M) or in the interior, but should be piecewise smooth and of dimension one less than Ω for flows, and of the same dimension as Ω for maps. Here we allow a particle to be injected at the hole, so it is absorbed only when it reaches the hole at strictly positive time. If we denote by Ω_t the subset of Ω that does not reach the hole by time t,

$$\Omega_t = \{x \in \Omega : \Phi^s(x) \notin H, \quad \forall s \in (0, t]\} \tag{1}$$

[6]That is, $\mu(\Phi^{-t}A) = \mu(A)$ for any $A \subseteq \Omega$ and any t (positive for noninvertible maps).

a typical question to ask is that given a set of initial conditions distributed according to some probability measure μ_0 at time $t = 0$, what is the probability $P(t) = \mu_0(\Omega_t)$ that the particle survives until time t? How does this probability behave as a function of t, the initial measure μ_0, the hole location H, its size relative to the billiard boundary $h = |H|/|M|$ (for general maps, $|H|/|\Omega|$), and the shape of the billiard $\mathcal{D}$?

In mathematical literature [51, 57] the term "open billiard" has been used to incorporate additional conditions. In terms of our notation the outer boundary of $\mathcal{D}$ is a strictly convex set forming the "hole" through which particles escape; $\mathcal{D}$ also excludes three or more strictly convex connected obstacles in its interior satisfying a non-eclipsing condition, that is, the convex hull of the union of any two obstacles does not intersect any other obstacles. This ensures that there is a trapped set of orbits that never escape, hyperbolic and with a Markov partition, leading to a relatively good understanding [35, 37, 51, 57, 65]. Billiards satisfying these conditions will be denoted here as "non-eclipsing" billiards.

For strongly chaotic systems (including the Anosov maps mentioned above and dispersing billiards with finite horizon discussed below) we expect that $P(t)$ decays with time as described by an (exponential) escape rate

$$\gamma = -\lim_{t\to\infty} \frac{1}{t} \ln P(t) \tag{2}$$

which exists and is independent of a reasonable class of initial measures μ_0.[7] We recall that t corresponds either to discrete or continuous time depending on the context, with (the more physical) continuous time implied for billiards except where otherwise stated.

We now discuss conditionally invariant measures; see also [28]. The renormalised evolution Φ_H^t of the initial measure μ_0 is defined by its action on sets $A \subseteq \Omega$:

$$\mu_t(A) = (\Phi_H^t \mu_0)(A) = \frac{\mu_0(\Phi^{-t}(A) \cap \Omega_t)}{\mu_0(\Omega_t)} \tag{3}$$

It is easy to check that Φ_H^t satisfies the semigroup property $\Phi_H^s \circ \Phi_H^t = \Phi_H^{s+t}$ for non-negative s and t. If μ_t approaches a limit μ_∞ at which Φ_H^t is continuous (in a suitable topology), this limit is conditionally invariant, ie $\Phi_H^t \mu_\infty = \mu_\infty$ for all $t \geq 0$, and gives the escape rate

$$\mu_\infty(\Omega_t) = e^{-\gamma t} \tag{4}$$

also for all $t \geq 0$.

[45] makes the important point that for open systems, conditionally invariant measures of Φ (projected onto M) and F, while supported on the same set, are generally not equivalent measures, even when the roof function τ is smooth, in constrast to the closed case. In other words escape properties of the flow do not follow trivially from those of the map. For billiard calculations, it is usually best to work first with the collision map F (projecting any interior holes onto the collision space M), then try to incoporate effects of the flow. For incorporation of flow effects in proofs of statistical properties of closed billiards, see [7, 20] and in calculation of escape rates and averages of open systems, see [16, 37, 48].

Discussion in [28] demonstrates that conditionally invariant measures can have properties similar to the SRB measures of closed hyperbolic systems, for example being smooth in the unstable direction. For non-invertible maps the measures satisfying the property of conditional invariance can however be highly non-unique.

The conditionally invariant measure μ_∞ is supported on the set of points with infinite past avoiding the hole, however most of these points reach the hole in the future. There is an invariant set of points never reaching the hole in past or future called the repeller; the natural invariant measure on this set is defined by the limit

$$\nu(A) = \lim_{t\to\infty} e^{\gamma t} \mu_\infty(A \cap \Omega_t) \tag{5}$$

[7] In [28] some examples are given where existence is shown for μ_0 equivalent to Lebesgue with density bounded away from 0 and ∞; this condition is probably too restrictive.

Pesin's formula generalises in the open case to the "escape rate formula" [28, 37]

$$\gamma = \sum_{i:\lambda_i>0} \lambda_i - h_{KS} \tag{6}$$

where the quantities on the right hand side are defined with respect to ν.[8] The escape rate formula has very recently been shown for a system that is not uniformly hyperbolic [12], however there is wide scope for further development both in maps without uniform hyperbolicity and in flows.

Open problem 1. *How general is the escape rate formula, Eq. (6)?*

There has been a very recent explosion of interest in the mathematics of open dynamical systems [1, 18, 27, 47, 57], again mostly restricted to strongly hyperbolic maps as above. In [18, 47] it is shown that γ/h can reach a limit, the local escape rate, as the hole shrinks to a point, and that the limit depends on whether the point is periodic. For example consider the map $f(x) = 2x \pmod 1$ and a point z of minimal period p so that $f^p(z) = z$, together with a sequence of holes H_n of size h_n and each containing z, with corresponding escape rates γ_n. Then there is a local escape rate

$$\lambda_z = \lim_{n\to\infty} \frac{\gamma_n}{h_n} = 1 - 2^{-p} \tag{7}$$

If the point z is aperiodic, then λ_z is equal to 1. For more general 1D maps, the 2^{-p} is replaced by the inverse of the stability factor $|(d/dx)f^p(x)|$ at $x = z$ and the invariant measure of the map (uniform in the above example) needs to be taken into account.

This and related results are, however available only for piecewise expanding maps and closely related systems, although "it is conceivable that in the near future this result could be applied e.g. to billiards." [47]. The very recent work [27] on the periodic Lorentz gas, a dispersing billiard model (see below) shows that γ (defined with respect to collisions, not continuous time) is well defined for sufficiently small holes, corresponding to a limiting conditionally invariant measure that is independent of the initial measure for a relatively large class of the latter, also that $\gamma \to 0$ as $h \to 0$, but does not say anything about γ/h. This leads to the open problem:

Open problem 2. *Local escape rate: Proof of a formula similar to Eq. (7) for sufficiently chaotic billiard models.*

Note that the holes typically considered in billiards, consisting of a set small in position but with arbitrary momentum,[9] include phase space distant from any short periodic orbit contained in the hole. Following the discussion in [16], the effects of the periodic orbit may then appear at a higher power of the hole size, compared to the above piecewise expanding map, so for example the formula might look something like $\gamma = h + h^2\gamma_2 + O(h^3)$ with γ_2 depending on properties of the shortest periodic orbit contained in the hole. Also note (see open problem 4 below) that if there is a local escape rate, P behaves like $e^{-\lambda_z ht}$, in other words like a function of the combination ht in this limit.

Another important question posed in this work is that of optimisation: how to choose a hole position to minimise or maximise escape, for example γ or even the whole function $P(t)$ [1, 18].

Open problem 3. *Optimisation: Specify where to place a hole to maximise or minimise a suitably defined measure of escape.*

[8] Proofs of the escape rate formula typically replace the sum of Lyapunov exponents by the (more general) Jacobian of the dynamics restricted to unstable directions. For two dimensional billiards, at most one Lyapunov exponent is positive.

[9] Note that other possibilities occur in applications (Sec. 5): Escape for trajectories with sufficiently small angle of incidence but at any boundary point is relevant to microlasers, and escape with probability depending on the boundary material is relevant to room acoustics.

In the above papers, slow escape is related to placing the hole on a short periodic orbit or on the part of phase space with the lowest "network load." The main question is how these criteria generalise to open dynamical systems (including billiards) with distortion, nonuniform hyperbolicity or no hyperbolicity.

As will become clear below, the existence of an exponential escape rate γ and nontrivial conditionally invariant measure μ_∞ is only expected in fairly special cases: there are many systems including billiards with slower (or occasionally faster) decay of the survival probability. For some systems with slower eventual decay of $P(t)$, a cross-over from approximately exponential decay at short times to algebraic decay at late times has been observed numerically [2]. We now turn to our main focus, that of open billiards, returning to general open dynamical systems in the applications section.

4 Open billiards

Now we consider open billiards, categorised according to dynamical properties of the corresponding closed case. The initial measure μ_0 for an open billiard is normally given by the equilibrium measure for the flow or Poincaré map, μ_Ω or μ_M respectively. An alternative is to consider injection at the hole by a restriction of the equilibrium measure, in other words transport through the billiard; in many cases this has little effect on the long time properties [2].

A few papers discussed here and in the statistical mechanics section below consider billiards on infinite domains, usually consisting of an infinite collection of non-overlapping obstacles. If both the billiard and any hole(s) are periodic, this is equivalent to motion on a torus; an example is [27]. However, if the billiard is aperiodic, or the hole is not repeated periodically, the above definitions break down because the equilibrium billiard measure is not normalisable. The above definition of ergodicity makes sense using an infinite invariant measure, but other properties including mixing and survival probability are problematic. Also, the Poincaré recurrence theorem fails and so recurrence properties need to be proved. One approach to defining $P(t)$ in the case of a non-repeated hole might be to choose an initial measure μ_0 with support in a bounded region, somewhat analogous to the finite (but rather arbitrary) outer boundary of the domain $\mathcal{D}$ chosen for non-eclipsing billiards above.

Integrable billiards A few billiards are integrable, ie perfectly regular, namely the ellipse, rectangle, equilateral triangle and related cases [67]. In the integrable circle with a hole of angle $2\pi h$ in the boundary [15], the leading order coefficient of $P(t)$ was obtained exactly; in particular the statement

$$\lim_{h\to 0}\lim_{t\to\infty} h^{\delta-1/2}(tP(t)-\frac{1}{\pi h})=0 \tag{8}$$

for all $\delta>0$ is equivalent to the Riemann Hypothesis, the greatest unsolved problem in number theory.[10] If instead both limits are taken simultaneously so that ht is constant, the survival probability numerically appears to reach a limiting function.

Open problem 4. *Scaling: For what billiards and in what limits does $P(t)$ reduce to a function of ht for a family of billiards with variable hole size h?*

Polygonal billiards The survey [40] gives a good introduction to polygonal billiards. A billiard collision involving a straight piece of boundary is equivalent to free motion in an "unfolded" billiard reflected across the boundary as in a mirror, using multiple Riemann sheets if the reflection leads to an overlap. General polygonal billiards with angles rational multiples of π can be unfolded

[10] The Riemann hypothesis is the statement that all the complex zeros of the Riemann zeta function, the analytic continuation of $\zeta(s)=\sum_{n=1}^{\infty} n^{-s}$ lie on the line $Re(s)=1/2$, and is related to the distribution of prime numbers [23].

into flat manifolds with nontrivial topology and conical singularities (from the corners), called translation surfaces, which are currently an active area of interest [5]. In such billiards each trajectory explores only a finite number of directions, however with this restriction is typically ergodic but not mixing [69]. There has been recent progress on showing recurrence for infinite rational polygonal billiards [68]

Very little is known about the case of irrational angles [43, 70], except that usual characteristics of chaoticity such as positive Lyapunov exponents are not possible here. We have the well known open question [62]:

Open problem 5. *Existence of periodic orbits: Do all triangular (or more generally polygonal) billiards contain at least one periodic orbit?*

Note that this admits arbitrary directions; it is easy to construct rational polygonal billiards for which a particular directional flow is never periodic. Also, there are a number of cases including rational and acute triangles for which periodic orbits can be explicitly constructed. Periodic orbits in polyhedra are considered in [10].

Another open problem has the interesting feature that mathematical literature conjectures that it is false [40], while physical literature conjectures that it is true [19]; recent work is found in [69]:

Open problem 6. *Mixing property: Is it possible for a polygonal billiard (necessarily with at least one irrational angle) to be mixing?*

There is very little work on open polygonal billiards given the paucity of knowledge about the closed case. In [30] it was found numerically that $P(t)$ decayed as C/t in an irrational polygonal billiard where at least one periodic orbit was present, and vanished at finite time in a directional flow for a rational polygonal billiard containing no periodic orbits. So we have

Open problem 7. *Is the asymptotic survival probability $P(t)$ as $t \to \infty$ in polygonal billiards entirely determined by the neighbourhoods of non-escaping periodic orbits?*

Dispersing billiards Curved boundaries of billiards lead to focusing (convergence) or dispersion (divergence) of an initially parallel set of incoming trajectories, depending on the sign of the curvature.[11] [12] We have already noted that one focusing billiard, the ellipse, is integrable; more generally all sufficiently smooth strictly convex billiards are not ergodic [50]. However there are a number of other classes of billiards which are not only ergodic, but have much stronger chaotic properties, to which we turn; [20] provides a good introduction and [64, 66] recent reviews.

Beginning with the Sinai billiard [63] which used a convex obstacle in a torus to ensure that all collisions were dispersing, there have been many new constructions of chaotic billiards and techniques to prove stronger statistical properties of existing billiards. For smooth dispersing cases such as Sinai's billiard, ergodic properties now include the Bernoulli property. The theory is technical due to singularities in the dynamics, requiring additional techniques for more difficult classes of singularity: All dispersing billiards have trajectories tangent to one or more parts of the boundary; perturbed trajectories either miss that part of the boundary entirely, or collide with an outgoing perturbation proportional to the square root of the incoming one, that is, infinite linear instability. In addition, the boundary may contain corners, in most cases leading to discontinuities in the dynamics. In the case of cusps (zero angle corners) the number of collisions near the cusp in a finite time is unbounded. A further complication occurs in motion on a torus if some trajectories

[11] Zero curvature, ie inflection points and similar, can be problematic; see footnote 4. The main exception is where part of the boundary is exactly flat. A class of billiards allowing some inflection points that has been studied in detail is that of semi-dispersing billiards [21, 66].

[12] In three or more dimensions the curvature at a boundary point generally depends on the plane of the incoming trajectory; this phenomenon is known as astigmatism and causes difficulties in the theory, hence many results are weaker and/or require additional assumptions; see for example [8, 57].

never collide ("infinite horizon condition"),[13] due to an accumulation of singularity sets near such orbits. Proof of exponential decay of correlations of the billiard map was possible using Young tower constructions of the late 1990s which generalise the concept of a Markov partition, but exponential decay of correlations for the flow is still conjectured; recent results for advanced statistical properties and treatment of continuous time may be found in [22, 7] and some open problems in [66].

An example of a dispersing billiard with corners is the diamond shown in Fig. 1. This was the numerical example in [16], where the exponential escape rate γ as above can be expanded in powers of the hole size h using series of correlation functions, as can the differential escape rate of one vs two holes. For example, the expression for the latter, following a derivation that is far from rigorous but consistent with numerical tests, is

$$\gamma_{AB} = \gamma_A + \gamma_B - \frac{1}{\langle\tau\rangle} \sum_{j=-\infty}^{\infty} \langle (u_A)(F^j \circ u_B)\rangle + \dots \tag{9}$$

where A and B label the holes, τ is the time from one collision to the next, u is a zero mean phase function (that is, $\langle u\rangle = 0$) equal to -1 on the relevant hole and $h\tau/\langle\tau\rangle$ elsewhere, angle brackets indicate integration with respect to normalised equilibrium measure on the billiard boundary μ_M, and F is the billiard map. The omitted terms have third or higher order correlations, and are expected to be of order h^3 as long as A and B do not contain points from the same short periodic orbit. Convergence of the sum over j follows from sufficiently fast decay of correlations, which do not need to be exponential. An extended diamond geometry was used as a model for heat conduction in [38].

Open problem 8. *Give a rigorous formulation of the escape rate of dispersing billiards with small holes along the lines of Eq. (9).*

Dispersing billiards, which include the non-eclipsing billiards defined above, also provide a likely setting for solving the open problems in Sec. 2.

Defocusing and intermittent billiards In addition to the dispersing billiards, strongly chaotic billiards may be constructed with the defocusing mechanism of Bunimovich, in which trajectories that leave a focusing piece of boundary have a sufficient time to defocus (ie disperse) before reaching another curved part of the boundary; for a recent discussion see [17]. In this case there are difficulties due to "whispering gallery" orbits close to a focusing boundary with finite but unbounded collisions in a finite time. Some concepts and results can be derived from the Pesin theory of nonuniformly hyperbolic dynamics [9, 20].

An example of a defocusing billiard is the stadium of Fig. 1; note that while it is ergodic and hyperbolic, the presence of "bouncing ball" orbits between the straight segments leads to intermittent quasi-regular behaviour, leading to some weaker statistical properties, so for example while the system is mixing, decay of correlations is now C/t for both the map and flow, leading to a non-standard central limit theorem with $\sqrt{n \ln n}$ (rather than $\sqrt{n}$) normalisation [6, 7].

The open stadium has been considered in [4, 32]; this is a good model for intermittency due to "bouncing ball" motions between the straight segments. The first paper discusses scaling behaviour associated with evolution of measures at long times and small angles close to the bouncing ball orbits, while the second finds an explicit expression for the leading coefficient of the survival probability

$$\lim_{t\to\infty} tP(t) = \frac{(3\ln 3 + 4)((a+h_1)^2 + (a-h_2)^2)}{4(4a+2\pi r)} \tag{10}$$

[13] In the context of infinite billiards, there has recently been proposed a "locally finite horizon condition", in which all straight lines pass through infinitely many scatterers, but the free path length is unbounded, see [68]. Here it is also assumed that the minimum distance between obstacles be bounded away from zero.

for a stadium with horizontal straight segments $x \in (-a, a)$ containing a small hole for $x \in (h_1, h_2)$ with $-a < h_1 < h_2 < a$, and semicircles of radius r. Note that this approaches a constant as $h \sim h_2 - h_1 \to 0$, thus demonstrating an example where fixed ht is *not* a correct scaling limit (compare open problem 4).

The stadium is the most famous defocusing billiard, but many of its properties are due to the intermittency arising from the bouncing ball orbits, or more generally a family of marginally unstable periodic orbits, rather than the defocusing mechanism per se; defocusing billiards need not have such orbits [17]. One source of interest and also difficulty with the stadium is the fact that orbits that leave the bouncing ball region are immediately reinjected back, with an angle that is approximately described using an independent stochastic process [4]. Other reinjection mechanisms are possible, depending on the properties of the curvature of the boundary approaching the end points of the bouncing ball orbits. In some cases small changes of the boundary of a stadium lead to a breakdown of ergodicity [39], which of course will also affect relevant escape problems.

Open problem 9. *Give a comprehensive characterisation of the dynamical (including escape) properties of marginally unstable periodic orbits in stadium-like billiards in terms of their reinjection dynamics.*

Mixed billiards Typical billiards are expected to have mixed phase space, that is, chaotic or regular depending on the initial condition, in a fractal hierarchy according to Kolmogorov-Arnold-Moser (KAM) theory [11]. Much remains to be understood about this generic case; it has been conjectured [2, 25] that in the case of area preserving maps the long time survival probability associated with stickiness near the elliptic islands decays as $t^{-\alpha}$ with a universal $\alpha \approx 2.57$.

Open problem 10. *Is there a universal decay rate in generic open billiards?*

One fruitful line of enquiry has been the introduction of mushroom billiards by Bunimovich; these have mixed phase space, but with a smooth boundary between the regular and chaotic regions. See Fig. 1 and [13]. The chaotic region of most of these is intermittent ("sticky") due to embedded marginally unstable periodic orbits [3], however an example of a non-sticky mushroom-like billiard was given in [13]. The decay of the survival probability in the case that the hole is in the chaotic region may be significantly slowed both by stickiness due to marginally unstable periodic orbits, and by the boundary itself.[14]

Open problem 11. *Characterise the escape properties of the boundary between regular and chaotic regions in mushroom and similar billiards.*

5 Physical applications

Statistical mechanics One important application of billiards is that of atomic and molecular interactions, for which steep repulsive potential energy functions can be approximated by the hard collisions of billiards. A system of many particles undergoing hard collisions corresponds to a high dimensional billiard. This type of billiard is described as semi-dispersing since during a collision of two particles the other particles are not affected, so there are many directions in the high dimensional collision space with zero curvature; however there has been recent progress in demonstrating ergodicity (conjectured by Boltzmann); see [21, 66]. Note, however, that ergodicity may be broken by arbitrarily steep potential energies approximating the billiard [61].

Open problem 12. *For physical systems with many particles which are predominantly chaotic, how prevalent and important are the regular regions ("elliptic islands") in phase space? [11, 13, 61].*

[14] Recent unpublished result of the author in collaboration with O. Georgiou.

Systems of many hard particles have also been instrumental in the discovery of a fascinating connection between microscopic and macroscopic dynamical effects, that of Lyapunov modes, although later found in systems with soft potentials; for a review see [74]. There are many other connections between dynamics in general and statistical mechanics [29, 54].

Low-dimensional billiards imitating hard-ball motion are popular for understanding statistical mechanics, particularly Lorentz gases consisting of an infinite number of convex (typically circular) scatterers and the Ehrenfest gases consisting of an infinite number of polygons [29, 37, 53]. Infinite periodic arrays in these models are equivalent to motion on a torus and can be treated using similar techniques to those of equivalent finite billiards [27, 38]. However, very little has been proven about the more physically realistic models with randomly placed non-overlapping obstacles [68]. There is numerical evidence for some statistical properties corresponding to a limiting Weiner (diffusion) process in some Ehrenfest gases [31, 43].

Open problem 13. *What are minimum dynamical properties required for a "chaotic" macroscopic limit, for example recurrence, ergodicity, a "normal" diffusion coefficient, a limiting Weiner process?*

While most of the above work pertains to closed systems (albeit on large or infinite domains), it is worth pointing out that the "escape rate formalism" [29, 37] relates escape rates in large open chaotic systems to linear transport properties; note that the escape rate is also related to other dynamical properties by Eq. (6). A final problem for this topic:

Open problem 14. *What experimental techniques might best probe the limits of statistical mechanics arising from the previous two open problems?*

Quantum chaos Billiards correspond to the classical (short wavelength) limit of wave equations for light, sound or quantum particles in a homogeneous cavity. The classical dynamics corresponds to the small wavelength, geometrical optics approximation. Semiclassical theory uses properties of this classical dynamics, especially periodic orbits, as the basis for a systematic treatment of the wave properties (eigenvalues and eigenfunctions of the linear wave operator). Comparisons are also made with the predictions of random matrix theory, in which the spacing of eigenvalues of many quantum systems follows universal laws based only on the chaoticity and symmetries of the problem [58].[15] Billiards are both necessary for calculating properties of wave systems and a testbed for semiclassical and random matrix theories. A useful survey of quantum chaos, including open quantum systems, is given in [55]; note that semiclassical approaches are also relevant to non-chaotic systems such as polygonal billiards [41].

An important recent development in open quantum systems has been the fractal Weyl conjecture relating the number of resonances up to a particular energy to fractal dimensions of classically trapped sets. Following [52] we recall that a quantum billiard is equivalent to the Helmholtz equation $(\nabla^2 + k^2)\psi = 0$ with a Dirichlet condition $\psi = 0$ at the boundary, the eigenvalues k are labelled k_n and are real. In this case Weyl's law gives a detailed statement to the effect that the number of states with $k_n < k$ is proportional to Vk^d in the limit $k \to \infty$ where d is the dimension of the space; effectively this means that each quantum state occupies the same classical phase space volume.

If the billiard is open, for example consisting of a finite region containing obstacles that do not prevent the particle escaping to infinity, the Helmholtz equation has resonance solutions where k has negative imaginary part. We have [52]

Open problem 15. *The fractal Weyl conjecture: The number of states with $Re(k_n) < k$ and $Im(k_n) > -C$ for some positive constant C is of order k^{d_H+1} where again $k \to \infty$ and d_H is the partial Hausdorff dimension of the non-escaping set.*

[15] There are fascinating conjectures [24] relating random matrix theory and the Riemann zeta function of footnote 10.

While open billiards of the non-eclipsing type have been under investigation from the beginning [65], the most progress to date has been in simpler systems such as quantum Baker maps [46, 56]. Recent work has also included smooth Hamiltonian systems [60] and optical billiards [73].

Experiments and further applications The case of microresonators where light is trapped by total internal reflection is of interest for practical applications in laser design; the hole then corresponds to a condition on momentum rather than position, and desired properties typically include large Q-factor (small imaginary part of k) so that the pumping energy required for laser action is small, together with a strongly directional wave function at infinity. The earliest example of a circular cavity has large Q-factor due to trapping of light by total internal reflection, but the symmetry precludes a directional output, so a number of efforts have been made to modify the circular geometry, using insight from classical billiards [3, 33, 59].

Other experiments of relevance have included microwaves, sound, atoms and electrons in cavities at scales ranging from microns to metres. These are aimed at demonstrating the capabilities of new experimental techniques, testing theoretical results from quantum chaos, and laying the groundwork for specific applications. Often modifications to the original billiard problem arise. For example, open semiconductor billiards can be constructed by confining a two dimensional electron gas (2DEG) using electric fields. In transport through such cavities, a weak applied magnetic field shifts the quantum phases (hence conductance) while the classical orbits are effectively unchanged, while a strong field also curves the classical orbits; also the walls are likely to be described by a somewhat soft potential energy function. For room acoustics, collisions involve a proportion of the sound escaping, being absorbed, being randomly scattered, and being reflected using the usual billiard law, depending on the materials at the relevant point on the boundary. Relevant references may be found in [16, 32, 42, 72].

Open dynamical systems that are more general than billiards are important for escape and transport problems in heat conduction [38], chemical reactions [36], astronomy [71] and nanotubes [43]. In these systems, open billiards provide a useful starting point for an understanding of more general classes of open dynamical systems.

Open problem 16. *Generalisations: Billiards with more realistic physical effects, for example soft wall potentials, external fields, dissipative and/or stochastic scattering (at the boundary or in the interior) or time dependent boundaries.*

The possibilities are endless; for this reason, it is increasingly important to carefully justify the mathematical and/or physical interest of any new model.

6 Discussion

We conclude with a discussion of some problems that draw together many classes of dynamics.

Open problem 17. *Exact expansions: For a given open (generalised) billiard problem, can the survival probability $P(t)$ be expressed exactly, or at least as an expansion for large t that goes beyond the leading term?*

The most likely candidates for this problem are those billiards that are best understood, the integrable and dispersing cases.

One feature that is common to all classes of billiards, classical and quantum is the role of periodic orbits in the open problem. Periodic orbits denote an exact recurrence to the initial state of a classical system, and so contribute directly to the set of orbits that enter from a hole and return there after one period. Periodic orbits may be of measure zero, but some positive measure neighbourhood of a periodic orbit will be sufficiently close as to have the same property. For (finite) billiards with no periodic orbits the Poincaré recurrence theorem guarantees similar behaviour.

For an initial measure supported other than on the hole, periodic orbits are relevant to the set of orbits which never escapes, and its neighbourhood often determines the long time survival probability. This is seen in many of the above sections, for integrable, other polygonal, chaotic and intermittent billiards. There is substantial existing theory for calculating the escape rate γ and other long time statistical properties (averages etc.) for hyperbolic systems, using either periodic orbits avoiding the hole [48] or passing through the hole [2], and also for semiclassical treatments of quantum systems [55].

Open problem 18. *Recurrence: Give a general description of how the escape problem is affected by recurrences in the system (periodic orbits or more general), in the vicinity of the hole and/or elsewhere, for systems with little or no hyperbolicity.*

Finally, a question posed in [15] but for which a systematic solution does not appear to exist in the literature:

Open problem 19. *Inverse problems: What information can be extracted about the dynamics of a billiard from escape measurements? Can you "peep" the shape of a drum? [16, 44].*

The open problems we have considered vary from the long standing and difficult (problem 5) to those arising very recently from active research that may as quickly solve them (problem 2). Even the simplest of integrable systems, the circle, has unexpected complexity, while a rigorous approach to hyperbolic billiards has a highly developed and technical theory, and the study of polygons with irrational angles is in its infancy. It is likely that progress will be made almost immediately on many of the problems in specific cases but that constructing reasonably complete solutions in general dynamical contexts will be a active area of research for many years. Finally, open billiards are interesting from a mathematical point of view and also required for solving practical problems; this would appear to be a particularly fruitful field for collaborations between mathematicians and physicists.

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